Lesson 16. Linear Programs in Canonical Form

0 Warm up

Example 1.

Let $A = \begin{pmatrix} 1 & 9 & 8 \\ 5 & 2 & 3 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Then $A\mathbf{x} =$ ______ .

1 Canonical form

- LP in **canonical form** with decision variables $x_1, \dots, x_n$:

$$\begin{aligned} \text{minimize / maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i && \text{for } i \in \{1, \dots, m\} \\ & x_j \geq 0 && \text{for } j \in \{1, \dots, n\} \end{aligned}$$

- In vector-matrix notation with decision variable vector $\mathbf{x} = (x_1, \dots, x_n)$:

$$\begin{aligned} \text{minimize / maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{CF}$$

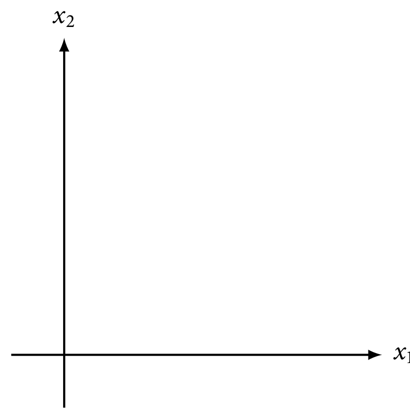
  - A has m rows and n columns, $\mathbf{b}$ has m components, and $\mathbf{c}$ and $\mathbf{x}$ each have n components

- We typically assume that $m \leq n$, and $\text{rank}(A) = m$

Example 2. Identify $\mathbf{x}$, $\mathbf{c}$, A, and $\mathbf{b}$ in the following canonical form LP:

$$\begin{aligned} \text{maximize} \quad & 3x + 4y - z \\ \text{subject to} \quad & 2x - 3y + z = 10 \\ & 7x + 2y - 8z = 5 \\ & x \geq 0, y \geq 0, z \geq 0 \end{aligned}$$

- A canonical form LP always has at least 1 extreme point (if it has a feasible solution)
  - Intuition: if solutions in the feasible region must satisfy $\mathbf{x} \geq \mathbf{0}$, then the feasible region must be "pointed"



2 Converting any LP to an equivalent canonical form LP

- Inequalities → equalities
  - **Slack** and **surplus** variables "consume the difference" between the LHS and RHS
  - If constraint i is a $\leq$-constraint, add a slack variable s_i:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \Rightarrow$$

  - If constraint i is a $\geq$-constraint, subtract a surplus variable s_i:

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad \Rightarrow$$

- Nonpositive variables → nonnegative variables
  - If $x_j \leq 0$, then introduce a new variable x'_j and substitute $x_j = -x'_j$ everywhere – in particular:

- Unrestricted ("free") variables → nonnegative variables
  - If x_j is unrestricted in sign, introduce 2 new nonnegative variables x_j^+, x_j^-
  - Substitute $x_j = x_j^+ - x_j^-$ everywhere
  - Why does this work?
    - Any real number can be expressed as the difference of two nonnegative numbers

Example 3. Convert the following LPs to canonical form.

$$\begin{array}{ll} \text{maximize} & 3x + 8y \\ \text{subject to} & x + 4y \leq 20 \\ & x + y \geq 9 \\ & x \geq 0,\ y \text{ free} \end{array} \qquad \begin{array}{ll} \text{minimize} & 5x_1 - 2x_2 + 9x_3 \\ \text{subject to} & 3x_1 + x_2 + 4x_3 = 8 \\ & 2x_1 + 7x_2 - 6x_3 \leq 4 \\ & x_1 \leq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{array}$$

3 Basic solutions in canonical form LPs

- Recall: a solution $\mathbf{x}$ of an LP with n decision variables is a **basic solution** if
  - (a) it satisfies all equality constraints
  - (b) at least n constraints are active at $\mathbf{x}$ and are linearly independent
- The solution $\mathbf{x}$ is a **basic feasible solution (BFS)** if it is a basic solution and satisfies all constraints of the LP
- What do basic solutions in canonical form LPs look like?

3.1 Example

- Consider the following canonical form LP:

$$
\begin{array}{llr}
\text{maximize} & 3x + 8y & \\
\text{subject to} & x + 4y + s_1 = 20 & (1)\\
 & x + y + s_2 = 9 & (2)\\
 & 2x + 3y + s_3 = 20 & (3)\\
 & x \geq 0 & (4)\\
 & y \geq 0 & (5)\\
 & s_1 \geq 0 & (6)\\
 & s_2 \geq 0 & (7)\\
 & s_3 \geq 0 & (8)
\end{array}
$$

- Identify the matrix A and the vectors $\mathbf{c}$, $\mathbf{x}$, and $\mathbf{b}$ in the above canonical form LP.

- Suppose $\mathbf{x}$ is a basic solution
  - How many linearly independent constraints must be active at $\mathbf{x}$?
  - How many of these must be equality constraints?
  - How many of these must be nonnegativity bounds?

- Let's compute the basic solution $\mathbf{x} = (x, y, s_1, s_2, s_3)$ associated with (1), (2), (3), (6), and (8)
  - It turns out that the constraints (1), (2), (3), (6), and (8) are linearly independent
  - Since the basic solution is active at the nonnegativity bounds (6) and (8),
  - The other variables, x, y, and s_2 are potentially nonzero
  - Substituting $s_1 = 0$ and $s_3 = 0$ into the other constraints (1), (2), and (3), we get

$$\begin{aligned} x + 4y + (0) \quad\quad\quad &= 20 \\ x + \;\; y \quad\quad + s_2 \quad &= 9 \\ 2x + 3y \quad\quad\quad + (0) &= 20 \end{aligned} \tag{$*$}$$

  - Let $\mathbf{x}_B = (x, y, s_2)$ and B be the submatrix of A consisting of columns corresponding to x, y, and s_2:

$$B = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

  - Note that ($*$) can be written as

$$B\mathbf{x}_B = \mathbf{b} \tag{$**$}$$

  - The columns of B linearly independent. Why?
  - ($**$) has a unique solution. Why?
  - It turns out that the solution to ($**$) is $\mathbf{x}_B = (4, 4, 1)$
  - Put it together: the basic solution $\mathbf{x} = (x, y, s_1, s_2, s_3)$ associated with (1), (2), (3), (6), and (8) is

4 Generalizing the example

- Now let's generalize what happened in the example above
- Consider the generic canonical form LP (CF)
  - Let n = number of decision variables
  - Let m = number of equality constraints
  - In other words, A has m rows and n columns
  - Assume $m \leq n$ and $\text{rank}(A) = m$
- Suppose $\mathbf{x}$ is a basic solution
  - How many linearly independent constraints must be active at $\mathbf{x}$?
  - Since $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$, how many nonnegativity bounds must be active?
- Generalizing our observations from the example, we have the following theorem:

Theorem 1. If $\mathbf{x}$ is a basic solution of a canonical form LP, then there exists m **basic variables** of $\mathbf{x}$ such that

(a) the columns of A corresponding to these m variables are linearly independent;
(b) the other $n - m$ **nonbasic variables** are equal to 0.

The set of basic variables is referred to as the **basis** of $\mathbf{x}$.

- Let's check our understanding of this theorem with the example
  - Back in the example, n = and m =
  - Recall that $\mathbf{x} = (x, y, s_1, s_2, s_3) = (4, 4, 0, 1, 0)$ is a basic solution
  - Which variables of $\mathbf{x}$ correspond to m LI columns of A?
  - Which $n - m$ variables of $\mathbf{x}$ are equal to 0?
  - The basic variables of $\mathbf{x}$ are
  - The nonbasic variables of $\mathbf{x}$ are
  - The basis of $\mathbf{x}$ is
- Let B be the submatrix of A consisting of columns corresponding to the m basic variables
- Let $\mathbf{x}_B$ be the vector of these m basic variables
- Since the columns of B are linearly independent, the system $B\mathbf{x}_B = \mathbf{b}$ has a unique solution
  - This matches what we saw in (∗∗) in the above example
- The m basic variables are potentially nonzero, while the other $n - m$ nonbasic variables are forced to be zero